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Flux Calculation at the Interface Between two Rock Types for Two-Phase Flow in Porous Media

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N° 2075
Octobre 1993

PROGRAMME 6

Calcul scientifique,
modélisation et
logiciels numériques

Rapport de recherche

1993

Flux calculation at the interface between two rock types for two-phase flow in porous media

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October 18, 1993

Abstract

When neglecting capillary effects, two-phase flow in porous media is modelled by a nonlinear hyperbolic equation. We consider the case where there is a discontinuity in the medium. Then the flux function of the conservation law is discontinuous with respect to the space variable, and we adress the problem of calculating a numerical flux at the interface between the two rock types. Several solutions are compared, one of them being justified mathematically.

Keywords : flow in porous media, conservation laws, nonlinear hyperbolic equations, discontinuous flux function, finite differences.

Calcul de flux à l'interface entre deux types de roches pour les déplacements diphasiques en milieu poreux

Résumé

En négligeant les effets capillaires, les déplacements diphasiques en milieu poreux sont modélisés par une équation hyperbolique non-linéaire. On considère le cas où le milieu présente une discontinuité. Alors la fonction de flux de la loi de conservation est discontinue par rapport à la variable d'espace, et on s'intéresse au problème du calcul du flux numérique à l'interface entre les deux types de roches. Plusieurs solutions sont comparées, l'une d'entre elles pouvant être justifiée mathématiquement.

Mots clés : Ecoulements en milieu poreux, lois de conservation, équations hyperboliques non-linéaires, fonction de flux discontinue, differences finies.

1 Introduction

Capillary-free two-phase incompressible flow is modelled by the following scalar nonlinear hyperbolic equation

$$\phi\frac{\partial S}{\partial t} + \frac{\partial f}{\partial x} = 0 \tag{1.1}$$

where ϕ is the porosity of the rock, $S = S_1$ is the saturation of phase 1 which lies in a bounded interval $[S_m, S_M]$:

$$0 \leq S_m \leq S \leq S_M \leq 1.$$

The flux function f is the Darcy velocity φ_1 of phase 1 and has the form

$$f = \varphi_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2}[q + (g_1 - g_2)\lambda_2]. \tag{1.2}$$

Here q denotes the total Darcy velocity

$$q = \varphi_1 + \varphi_2$$

where $\varphi_\ell, \ell = 1, 2$, denotes the Darcy velocity of phase ℓ with, for the second phase,

$$\varphi_2 = \frac{\lambda_2}{\lambda_1 + \lambda_2}[q + (g_2 - g_1)\lambda_1]. \tag{1.3}$$

Since the flow is assumed to be incompressible, the total Darcy velocity q is independent of the space variable x.

The quantities λ_ℓ, $\ell = 1, 2$ may be called effective mobilities. There are products of the absolute permeability K by the mobilities k_ℓ :

$$\lambda_\ell = K k_\ell, \ \ell = 1, 2.$$

The absolute permeability K may depend on x and the quantities k_ℓ and λ_ℓ are functions of S which satisfy the following properties :

$$\begin{array}{l} k_1 \text{ and } \lambda_1 \text{ are increasing functions of } S, \ k_1(S_m) = \lambda_1(S_m) = 0, \\ k_2 \text{ and } \lambda_2 \text{ are decreasing functions of } S, \ k_2(S_M) = \lambda_2(S_M) = 0. \end{array} \tag{1.4}$$

We also shall assume that these functions are smooth functions of the saturation S and so is the flux function f.

The gravity constants g_ℓ, $\ell = 1, 2$ of the phases are

$$g_\ell = g\rho_\ell\frac{dx}{dz}, \ \ell = 1, 2,$$

with g the acceleration due to gravity, ρ_ℓ the density of phase ℓ and z is the vertical position of the point of abscissa x.

The most popular way to discretize equation (1.1) is to use a cell-centered finite difference (or finite volume) method

$$\frac{\phi_i}{\Delta t}(S_i^{n+1} - S_i^n) + \frac{1}{h}(f_{i+1/2}^n - f_{i-1/2}^n) = 0, \tag{1.5}$$

where i denotes the cell index, $i + 1/2$ and $i - 1/2$ are indices for the end points of cell i, Δt and h are the time and space discretization intervals which satisfy a CFL condition.

The numerical flux at the interface between two cells is a function of the lefthand and righthand values of the saturation :

$$f_{i+1/2} = F(S_i, S_{i+1}).$$

It is calculated by solving exactly or approximately the Riemann problem, denoted by (S_i, S_{i+1}, f), associated with the initial data S_i, S_{i+1} and the (continuous) flux function f. For the scheme to be convergent, the function F must be consistent, monotone and Lipshitz continuous [7, 3].

In the following, we will consider two numerical flux functions. The Godunov flux is defined by

$$F^G(a,b) = \begin{cases} \min\limits_{[a,b]} f & \text{if } a < b, \\ \max\limits_{[a,b]} f & \text{if } a \geq b. \end{cases} \tag{1.6}$$

and is obtained by solving exactly the Riemann problem[6]. Here and throughout the rest of this paper a and b will denote respectively the lefthand and the righthand values of the saturation.

The other numerical flux function that we consider is the upstream mobilities flux. It is an ad-hoc flux for two-phase flow in porous media, invented by petroleum engineers from simple physical considerations. It is given by the following formula :

$$\begin{aligned} F^{UM}(a,b) &= \frac{\lambda_1^*}{\lambda_1^* + \lambda_2^*}[q + (g_1 - g_2)\lambda_2^*], \\ \lambda_\ell^* &= \begin{cases} \lambda_\ell(a) & \text{if } \varphi_\ell > 0, \\ \lambda_\ell(b) & \text{if } \varphi_\ell \leq 0. \end{cases} \end{aligned} \tag{1.7}$$

As we can see, the flux is calculated using the mobilities of the phases which are upstream with respect to the flow of the phases. This flux has been shown to have all the desired properties for convergence of the finite difference scheme : it is monotone, consistent and Lipshitz-continuous [8, 1].

2 The case of two rock types : an intuitive solution

In real life, the reservoir or the core sample may not be homogeneous and the domain of calculation may be divided in several regions with different rock types, the physical constants and the functions of saturation being different from one rock type to the other.

In the following, we consider a one-dimensional domain divided into two regions, rock type I on the left and rock type II on the right (see fig. 2.1). We also assume that the boundary between the two rock types coincides with a discretization point. The discontinuities in the physical constants and in the mobilities result in a flux function f which is discontinuous from one cell to the other. However since the mobilities satisfy properties (1.4), the two flux functions associated with the two rock types take the same values at their end points (see fig. 2.2):

$$f^{\mathrm{I}}(S_m^{\mathrm{I}}) = f^{\mathrm{II}}(S_m^{\mathrm{II}}) = 0, \quad f^{\mathrm{I}}(S_M^{\mathrm{I}}) = f^{\mathrm{II}}(S_M^{\mathrm{II}}) = q. \tag{2.1}$$

We will use this property when defining a numerical flux below.

Rock type I	**Rock type II**
$\phi^{\mathrm{I}},\ K^{\mathrm{I}},\ k_1^{\mathrm{I}},\ k_2^{\mathrm{I}}$	$\phi^{\mathrm{II}},\ K^{\mathrm{II}},\ k_1^{\mathrm{II}},\ k_2^{\mathrm{II}}$
$S_m^{\mathrm{I}} \leq S \leq S_M^{\mathrm{I}}$	$S_m^{\mathrm{II}} \leq S \leq S_M^{\mathrm{II}}$
$f \equiv f^{\mathrm{I}}$	$f \equiv f^{\mathrm{II}}$

Figure 2.1: Constants, mobilities and fluxes for two rock types

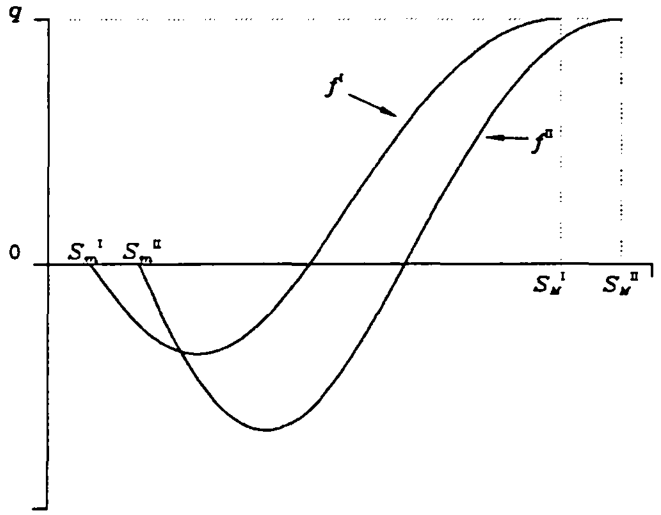

Figure 2.2: Two flux functions at the interface between two rock types

At the interface between two rock types, the saturation may be discontinuous but the Darcy velocities must be continuous in order to preserve the mass balance of the two phases.

When thinking in terms of Godunov fluxes, formulas (1.6) cannot be applied at the interface between two rock types since we do not know which flux function to use to calculate the Godunov flux. So the problem is now to define there a numerical flux.

A solution was given in [2] from the following considerations. Introduce a saturation $\overline{S}$ in an infinitely small transition zone (see fig. 2.3). Solving the Riemann problem in the domain with rock type I would give a numerical flux $F^{G,\mathrm{I}}(a,\overline{S})$ and solving the Riemann problem in the domain with rock type II would give a numerical flux $F^{G,\mathrm{II}}(\overline{S},b)$. Since the Darcy velocity must be continuous, we are looking for a number $F^G(a,b)$ such that

$$F^G(a,b) = F^{G,\mathrm{I}}(a,\overline{S}) = F^{G,\mathrm{II}}(\overline{S},b). \tag{2.2}$$

The existence of such a number is proven by the following theorem.

<u>Theorem</u> **2.1** *There exists a unique number $F^G(a,b)$ satisfying equation (2.2).*

<u>Proof</u>: First, since F^G is a monotone flux, $F^{G,\mathrm{I}}(a,S)$ is decreasing and $F^{G,\mathrm{II}}(S,b)$ is increasing with respect to S. Then it remains to show that the ranges of the functions $F^{G,\mathrm{I}}(a,\cdot)$ and

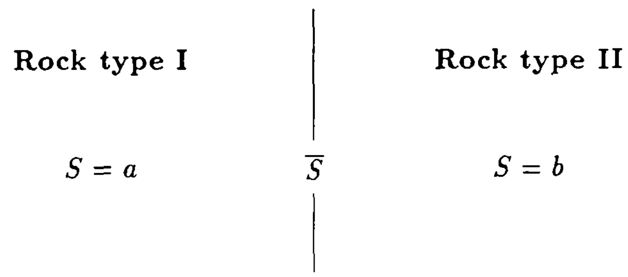

Figure 2.3: The saturation $\overline{S}$ in the transition zone

$F^{G,\mathrm{II}}(\cdot, b)$ have a nonempty intersection. From the definition of Godunov fluxes (1.6) we have

$$F^{G,\mathrm{I}}(a,S) \in [\min_{[a,S_M^{\mathrm{I}}]} f^{\mathrm{I}}, \max_{[S_m^{\mathrm{I}},a]} f^{\mathrm{I}}\,], \quad F^{G,\mathrm{II}}(S,b) \in [\min_{[S_m^{\mathrm{II}},b]} f^{\mathrm{II}}, \max_{[b,S_M^{\mathrm{II}}]} f^{\mathrm{II}}\,].$$

Since the two flux functions have the same endpoints (see equalities (2.1)), the bounds of their ranges satisfy

$$\begin{array}{ccccccccc} \min\limits_{[a,S_M^{\mathrm{I}}]} f^{\mathrm{I}} & \le & f^{\mathrm{I}}(S_M^{\mathrm{I}}) & = & q & = & f^{\mathrm{II}}(S_M^{\mathrm{II}}) & \le & \max\limits_{[b,S_M^{\mathrm{II}}]} f^{\mathrm{II}}, \\ \min\limits_{[S_m^{\mathrm{II}},b]} f^{\mathrm{II}} & \le & f^{\mathrm{II}}(S_m^{\mathrm{II}}) & = & 0 & = & f^{\mathrm{I}}(S_m^{\mathrm{I}}) & \le & \max\limits_{[S_m^{\mathrm{I}},a]} f^{\mathrm{I}}. \end{array}$$

Therefore their ranges have a nonempty intersection.

Fig. 2.4 shows how to calculate the numerical flux $F^G(a,b)$ in the case where the flux functions are not monotonous, as when gravity effects are large. In the simpler case where the flux functions are monotonous, say increasing, as when the phases are both flowing in the same direction, it is easy to check that $F^G(a,b) = f^{\mathrm{I}}(a)$, that is the numerical flux is the upstream flux.

Remark 1 *The value taken by the functions $F^{G,\mathrm{I}}(a,\cdot)$ and $F^{G,\mathrm{II}}(\cdot,b)$ when their graph intersect is unique, but the corresponding value of the saturation $\overline{S}$ may not be unique.*

Remark 2 *The construction above is valid for any smooth flux function as long as equation (2.1) holds. Therefore it can be applied to conservation laws which may arise from other physical problems than two-phase flow in porous media.*

Remark 3 *One checks easily that when $f^{\mathrm{I}} \equiv f^{\mathrm{II}}$ (continuous case) the above calculation gives the standard Godunov flux.*

3 Interpretation as a Riemann problem

The calculation of the numerical flux proposed in the previous section has been obtained from heuristic considerations. However the Riemann problem with a discontinuous flux function has been independently studied in [4] and the Cauchy problem is studied in [5].

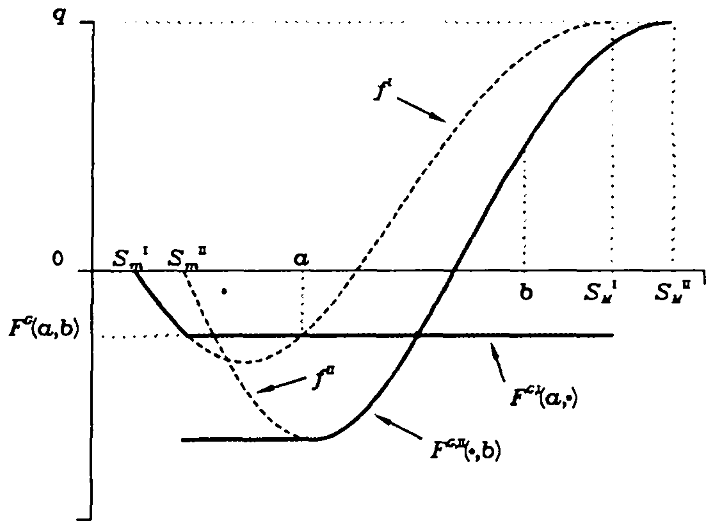

Figure 2.4: Calculation of the numerical flux $F^G(a,b)$ as the value at the intersection of the functions $F^{G,\mathrm{I}}(a,\cdot)$ and $F^{G,\mathrm{II}}(\cdot,b)$

In order to construct the solution of the Riemann problem, in [4] the authors consider the two Riemann problems (a,S,f^{I}) and (S,b,f^{II}), and introduce the sets

$$H^{\mathrm{I}}(a) = \{S;\ (a,S,f^{\mathrm{I}}) \text{ is solved by waves of nonpositive speed only}\}$$
$$H^{\mathrm{II}}(b) = \{S;\ (S,b,f^{\mathrm{II}}) \text{ is solved by waves of nonnegative speed only}\}.$$

Since $[S^{\mathrm{I}}_m, S^{\mathrm{I}}_M] \cap H^{\mathrm{I}}(a)$ and $[S^{\mathrm{II}}_m, S^{\mathrm{II}}_M] \cap H^{\mathrm{II}}(b)$ are closed sets, one may define

$$c^{\mathrm{I}}(a,S) = \{S' \in H^{\mathrm{I}}(a); |S-S'| \,\text{minimum}\}\,, \quad h^{\mathrm{I}}(a,S) = f^{\mathrm{I}}(c^{\mathrm{I}}(S))\,,$$
$$c^{\mathrm{II}}(b,S) = \{S' \in H^{\mathrm{II}}(b); |S-S'| \,\text{minimum}\}\,, \quad h^{\mathrm{II}}(b,S) = f^{\mathrm{II}}(c^{\mathrm{II}}(S)).$$

The relationship of these constructions with that of section 2 is given by the following equalities which can be easily checked :

$$F^{G,\mathrm{I}}(a,\cdot) \equiv h^{\mathrm{I}}(a,\cdot)\,, F^{G,\mathrm{II}}(\cdot,b) \equiv h^{\mathrm{II}}(b,\cdot)\,.$$

Therefore the construction described in section 2 gives a flux which is that given by the exact solution to the Riemann problem. It is obtained by piecing together waves travelling to the left produced by the lefthand side Riemann problem (a,S,f^{I}) and waves travelling to the right produced by the rightthand side Riemann problem (S,b,f^{II}). In other words we remove the waves crossing the interface between the two rock types, replacing them by waves with zero speed.

4 The upstream mobilities numerical flux

The same method as described in section 2 for the Godunov flux can be applied to the upstream mobilities flux (1.7). Again introducing an infinitely small transition zone where the saturation is $\overline{S}$ we look for a number $F^{UM}(a,b)$ such that

$$F^{UM}(a,b) = F^{UM,\mathrm{I}}(a,\overline{S}) = F^{UM,\mathrm{II}}(\overline{S},b). \tag{4.1}$$

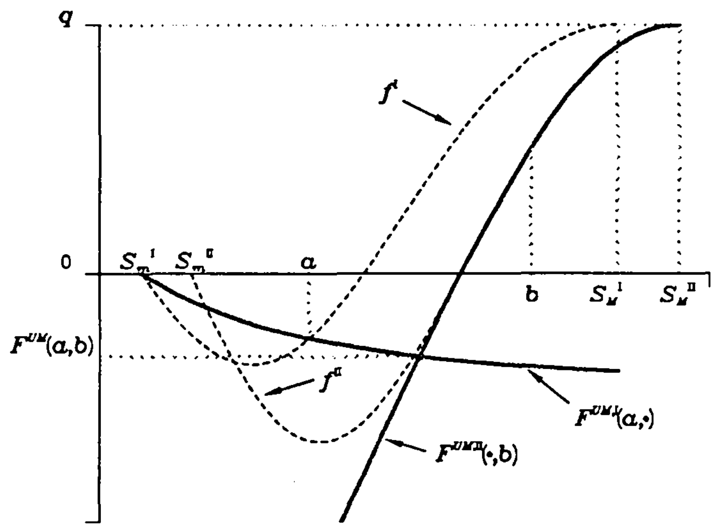

Figure 4.1: Calculation of the numerical flux $F^{UM}(a,b)$ as the value at the intersection of the functions $F^{UM,\mathrm{I}}(a,\cdot)$ and $F^{UM,\mathrm{II}}(\cdot,b)$

Existence and uniqueness of such a number is given by the following theorem, similar to theorem 2.1

<u>Theorem</u> **4.1** *There is a unique number $F^{UM}(a,b)$ satisfying (4.1).*

<u>Proof</u> : Again, since F^{UM} is a monotone flux, $F^{UM,\mathrm{I}}(a,S)$ is decreasing and $F^{UM,\mathrm{II}}(S,b)$ is increasing with respect to S. Then it remains to check that the ranges of the functions $F^{UM,\mathrm{I}}(a,\cdot)$ and $F^{UM,\mathrm{II}}(\cdot,b)$ have a nonempty intersection. There are many cases to consider. To not bore the reader, let us consider only the case of fig. 4.1. Then the ranges of the functions $F^{UM,\mathrm{I}}(a,\cdot)$ and $F^{UM,\mathrm{II}}(\cdot,b)$ are the following intervals :

$$\begin{aligned} F^{UM,\mathrm{I}}(a,S) &\in \quad [\,0\,,\ \frac{\lambda_1^{\mathrm{I}}(S_M^{\mathrm{I}})}{\lambda_1^{\mathrm{I}}(S_M^{\mathrm{I}})+\lambda_2^{\mathrm{I}}(a)}(q+(g_1-g_2)\lambda_2^{\mathrm{I}}(a))]\,, \\ F^{UM,\mathrm{II}}(S,b) &\in \quad [\,\frac{\lambda_1^{\mathrm{II}}(b)}{\lambda_1^{\mathrm{II}}(b)+\lambda_2^{\mathrm{II}}(S_m^{\mathrm{II}})}(q+(g_1-g_2)\lambda_2^{\mathrm{II}}(S_m^{\mathrm{II}}))\,,\ q\,]. \end{aligned}$$

Since λ_2^{II} is a decreasing function and since the quantity $q+(g_1-g_2)\lambda_2^{\mathrm{II}}(S)$ is negative for small value of S, we have

$$\frac{\lambda_1^{\mathrm{II}}(b)}{\lambda_1^{\mathrm{II}}(b)+\lambda_2^{\mathrm{II}}(S_m^{\mathrm{II}})}[q+(g_1-g_2)\lambda_2^{\mathrm{II}}(S_m^{\mathrm{II}})] \le f^{\mathrm{II}}(S_m^{\mathrm{II}}) = 0$$

which implies that the ranges of the two functions have a nonempty intersection.

On one hand calculation of the flux given by equation (4.1) is fairly complicated to implement and there is no advantage in it compared to that given by (2.2) using the Godunov flux. On the other hand, even though there is no mathematical support for it in the case of two rock types, there is an easy way to adapt the upstream mobility flux defined by equations (1.7) to this case :

$$
\begin{aligned}
&F^{UM}(a,b) = \frac{\lambda_1^*}{\lambda_1^* + \lambda_2^*}(q + (g_1 - g_2)\lambda_2^*), \\
&\lambda_\ell^* = \begin{cases} \lambda_\ell^{\mathrm{I}}(a) & \text{if } \varphi_\ell > 0, \\ \lambda_\ell^{\mathrm{II}}(b) & \text{if } \varphi_\ell \le 0, \end{cases} \qquad \ell = 1, 2.
\end{aligned}
\tag{4.2}
$$

This is the way that petroleum engineers usually implement the calculation of the upstream mobility in their reservoir simulation codes when different rock types are present.

5 Numerical experiments

In this section we compare experimentally three flux calculations of practical interest : that based on the exact solution (2.2) of the Riemann problem, the upstream mobility flux (4.2) and a third flux that one would naturally want to try. This third flux is the regular Godunov flux for the flux function obtained by averaging the righthand and the lefthand flux functions at the interface between the two rock types (assuming that they have the same domain).

We consider an idealized experiment with a closed core set vertically, initially saturated with the heavy fluid in the upper half and with the light fluid in the lower half. In the upper half there is one rock type with absolute permeability K_T and in the lower half with higher permeability K_B (see fig. 5.1). During the experiment the heavy phase moves downward while the light phase moves upward.

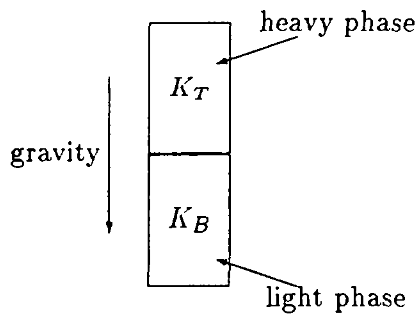

Figure 5.1: An heteregeneous vertical core with permeability K_T in the top half and K_B in the bottom half. At initial time, the top half is saturated with the heavy phase and the bottom part with the light phase.

The data that are used are the following :

$$
\begin{aligned}
&\phi = 1,\ q = 0,\ g_1 = 1,\ g_2 = 2,\ \lambda_1 = \tfrac{1}{2}KS^2,\ \lambda_2 = \tfrac{1}{2}K(1-S)^2, \\
&K = K_T \text{ if } x < 0 \text{ and } K = K_B \text{ if } x < 0.
\end{aligned}
$$

In the followings figures, the saturation of the heavy phase is shown at $t = 2$.

In figure 5.2 we present first in the homogeneous case when $K_T = K_B = 1$. We observe, as expected, that the scheme using the Godunov flux gives results which converge faster than when using the upstream mobility flux. However the latter gives also fairly good results, but the Courant-Friedrich condition requires time steps which are about two thirds of those required when using the Godunov flux.

In figures 5.3 and 5.4, we increase the absolute permeability to $K_B = 10$ and $K_B = 1000$. One notices that whatever the flux calculations, all curves converge to the same solution. That corresponding to the exact Riemann solver are the fastest to converge. Even though there is no mathematical justification for the two other methods, there are also converging : that using an average permeability with the Godunov flux are converging more slowly and that using upstream mobilities are the slowest to converge. Therefore they may be considered as approximate Riemann solvers.

The choice of the method depends on how much accuracy one wants. If one needs a very accurate solution, the exact Riemann solver should be used. However this is not the case, most of the time, when simulating flow in porous media, since there are so many physical incertainties ; so the upstream mobility flux, which is easier to implement and less costly, is usually sufficient.

Finally it should be pointed out that when there is a large variation in absolute permeabilities, it implies the same variation in the Courant-Friedrich condition. Therefore one should use some local time stepping procedure which allows for small time stepping in regions where the absolute permeability is large and large time steps in regions where the absolute permeability is small, in order to avoid to have to use the smallest time step in the whole domain.

6 Conclusion

We showed how to calculate the numerical flux at the interface between two rock types, corresponding to the exact solution of the associated Riemann problem.

This numerical flux has been compared, in a finite difference calculation, to the upstream mobility flux that is commonly used by petroleum engineers. It is more complicated to implement but it gives more accurate results.

The upstream mobility flux, though not justified mathematically, gives results that also converge to the right solution and therefore can be viewed as a numerical flux associated with an approximate Riemann solver.

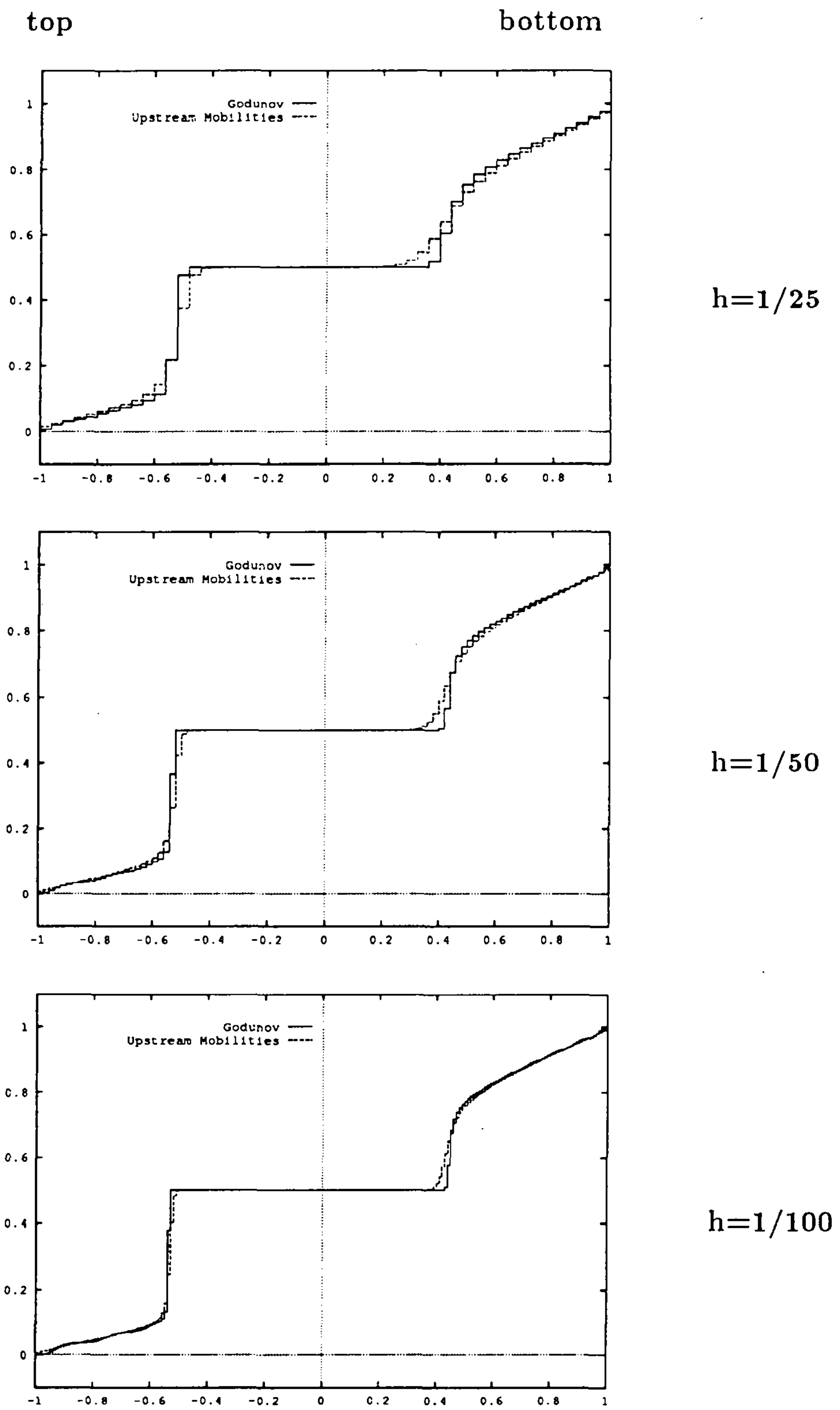

Figure 5.2: Comparaison of solutions obtained with the Godunov flux and with the upstream mobilities flux when $K_T = K_B = 1$ (homogeneous case).

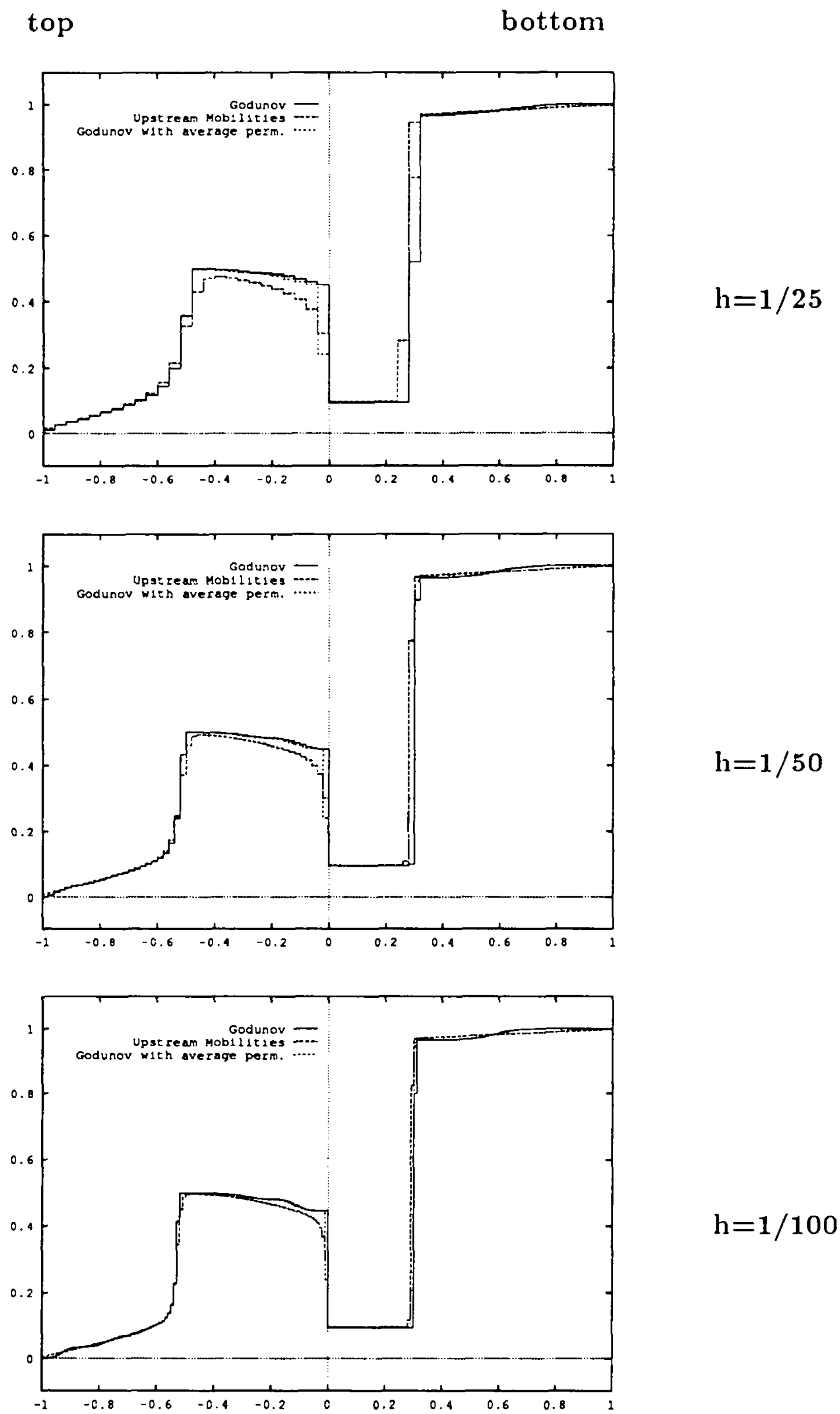

Figure 5.3: Comparaison of solutions obtained with the Godunov flux using the method of section 2 or an average permeability, and with the upstream mobilities flux when $K_T = 1, K_B = 10$.

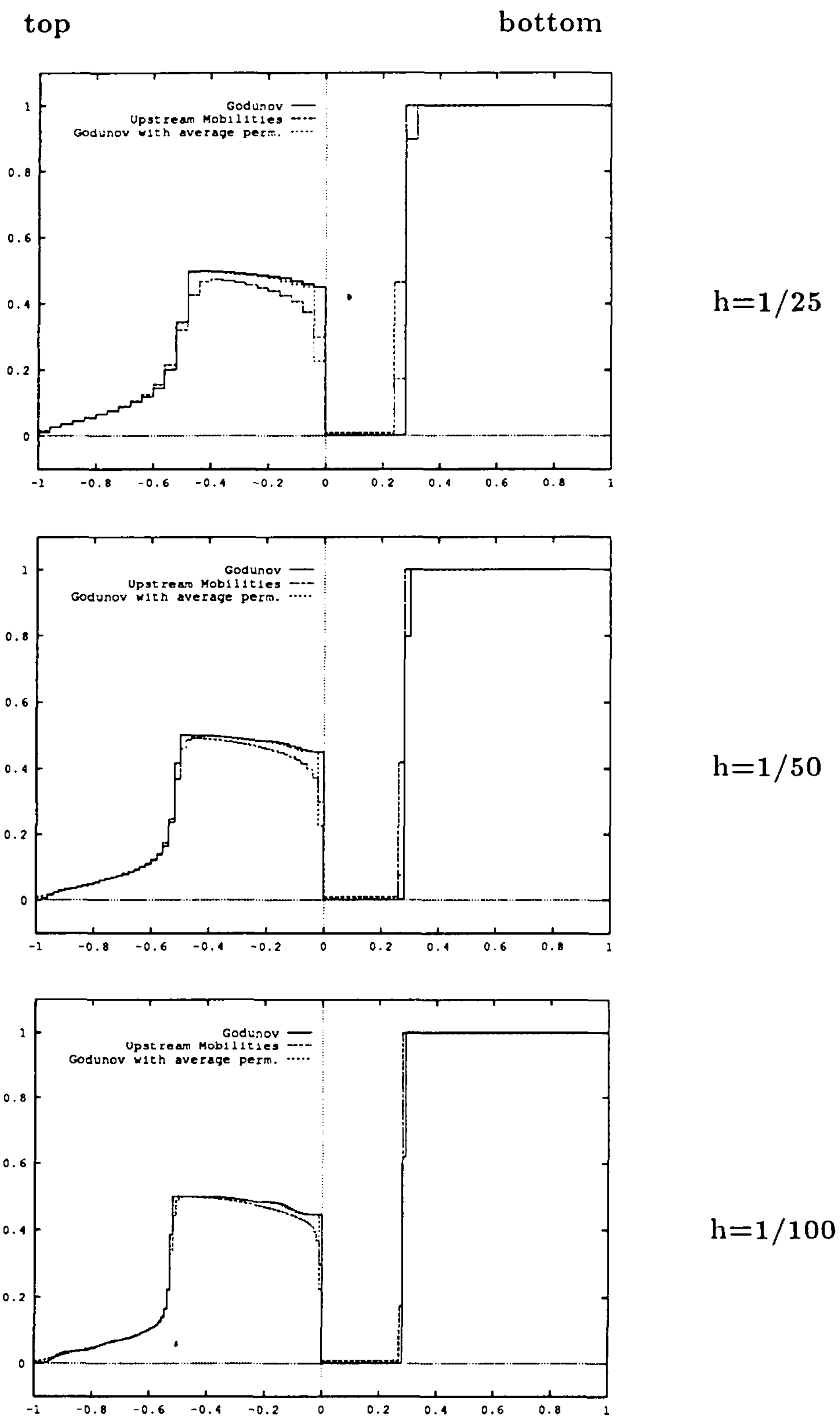

Figure 5.4: Comparaison of solutions obtained with the Godunov flux using the method of section 2 or an average permeability, and with the upstream mobilities flux (dotted line) when $K_T = 1, K_B = 1000$.

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ISSN 0249 - 6399

RR.2875